Mathematics of Growth and Human Population
Growth and Population

• Growth Rate
  → Exponential Growth
  → Half-life and Doubling Times
  → Disaggregated Growth

• Resource Consumption

• Logistic and Gaussian Growth Models

• Human Population Growth
  → Birth, Death, Fertility Rates
  → Age Structures


Growth Rate

• Growth rate enables prediction of future sizes—important for decisionmaking
  → Fuel usage and air pollution
  → Improvements in energy efficiency
  → Population growth and water demand
  → Deforestation rates and global effects
  → Cost and clean-up time of accidental contamination
Exponential Growth

Growth rates are proportional to the present quantity of people, resources, etc.

Example: Number of students in a school increases by 2% each year.

\[ N_0 = \text{starting number of students} \]
\[ N_t = \text{number of students in } t \text{ years} \]
\[ r = \text{annual growth rate} \]
Year zero = \( N_0 \)
Year one = \( N_1 = N_0 + rN_0 = N_0(1 + r) \)
Year two = \( N_2 = N_1 + rN_1 = N_1(1 + r) = N_0(1 + r)^2 \)
Year three = \( N_3 = N_2(1 + r) = N_0(1 + r)^3 \)

...Year \( t = N_t = N_0(1 + r)^t \)
Year \( t = N_t = N_0(1 + r)^t \)

Exponential law for periodic increments of growth

— discrete increases at the end of each time period

Example

If the school has 1500 students now and the Board of Education decides to increase the student body by 2% every fall, how many students will there be in 7 years?
Year $t = N_t = N_0(1 + r)^t$

$t = 7 \text{ yr}$
$N_0 = 1500 \text{ students}$
$r = 0.02$

Year 7 $= N_7 = 1500(1 + 0.02)^7$

$= 1723 \text{ students}$

A more realistic model uses *continuous* growth over time, with the growth rate again proportional to population size $N$ ...
$r$ is the growth rate with units $1/time$

$$\frac{dN}{dt} = r \times N$$

$$N = N_0 e^{rt}$$
Example

From 1990 to 1997, installed wind power in the U.S. grew by about 0.15% per year, resulting in a wind capacity of 1700 MW in 1997. If these rates are sustained, what will the wind energy capacity be in 2008?

\[ N = N_0 e^{rt} \]

\[ N_0 = 1700 \text{ MW}, \quad r = 0.0015 \text{ yr}^{-1}, \quad t = 11 \text{ years} \]

\[ N = 1700 \text{ MW} \times e^{0.0015 \text{ yr}^{-1} \cdot 10 \text{ yr}} = 1728 \text{ MW} \]
## Total Energy Consumption (U.S.)

<table>
<thead>
<tr>
<th></th>
<th>2008</th>
<th>2035</th>
<th>2008–2035 growth rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Petroleum (10^6 bbl/day)</td>
<td>20</td>
<td>22</td>
<td>0.4%</td>
</tr>
<tr>
<td>Natural Gas (10^{12} ft³)</td>
<td>23.2</td>
<td>26.5</td>
<td>0.5%</td>
</tr>
<tr>
<td>Coal (10^{15} BTU)</td>
<td>22.4</td>
<td>24.3</td>
<td>0.3%</td>
</tr>
<tr>
<td>Nuclear (10^9 kW-hr)</td>
<td>806</td>
<td>874</td>
<td>0.3%</td>
</tr>
<tr>
<td>Hydropower and other renewables (10^{15} BTU)</td>
<td>7.0</td>
<td>11.8</td>
<td>1.9%</td>
</tr>
</tbody>
</table>

Source: www.eia.gov
Actual growth of wind energy from 1998 to 2003 was about 23.1%, with total capacity in 2003 of 6374 MW. What is the new estimate of wind energy capacity in 2008?

\[ N = 6374 \, \text{MW} \times e^{(0.231 \, \text{yr}^{-1} \times 5 \, \text{yr})} = 20231 \, \text{MW} \]

19549 MW this June; 9022 MW under construction
Example

Decay instead of growth—

If the present mountain lion habitat in California is 8400 km², and is shrinking by 5% per year, when will it reach 5000 km²?

\[ N = N_0 e^{rt} \]

\[
5000 \text{ km}^2 = 8400 \text{ km}^2 e^{(-0.05 \text{ yr}^{-1})t}
\]

\[
\ln \left( \frac{5000}{8400} \right) = \frac{t}{-0.05 \text{ yr}^{-1}}
\]

\[
t = \frac{\ln \left( \frac{5000}{8400} \right)}{-0.05 \text{ yr}^{-1}} = 10.4 \text{ yr}
\]
Doubling Times and Half-life

\( t_d = \text{time for population to double} \)

\( t_{1/2} = \text{time for half of population to decay away} \)

\[ N = N_0 e^{rt} \]

Grow from \( N_0 \) to \( 2N_0 \):

\[ 2N_0 = N_0 e^{rt} \Rightarrow 2 = e^{rt} \Rightarrow \ln(2) = rt \]

\[ t_d = \frac{\ln(2)}{r} = \frac{0.692}{r} \]

\[ = \frac{70}{r(\%)} \]
If it took 300 years for the world’s population to increase from 0.5 billion to 4 billion and we assume exponential growth over that time period, what is the growth rate?

\[ N = N_0 e^{rt} \]

\[ r = \frac{\ln \left( \frac{N}{N_0} \right)}{t} \]

\[ r = 6.93 \times 10^{-3} \text{ yr}^{-1} \]

\[ t_d = \frac{\ln(2)}{r} = \frac{0.692}{r} \]

\[ 0.5 \rightarrow 1.0 \rightarrow 2.0 \rightarrow 4.0 \]

Took 300 years to double three times, so \( t_d = 100 \text{ yr} \)

\[ r = 6.92 \times 10^{-3} \text{ yr}^{-1} \]
Half-life

Switch to a decay situation where we go from $N_0$ to $0.5N_0$

$$N = N_0 e^{-kt}$$

$$N = \frac{1}{2} N_0 = N_0 e^{-kt}$$

$$t_{1/2} = \frac{\ln \left( \frac{1}{2} \right)}{-k} = \frac{\ln (1) - \ln (2)}{-k} = \frac{0.692}{k}$$

“Lifetime” is similar to half-life, except it’s how long it takes to decrease by 2.71 (e):

$$\tau = \frac{1}{k}$$
Example

A lake was impacted by a benzene spill of 100 kg. Benzene volatilizes and biodegrades, and after 5 days, there are 75 kg left.

When will half of the benzene be gone?

First find \( k \), then get \( t_{1/2} \)

\[
N = N_0 e^{-kt}
\]

75 kg = 100 kg \( \cdot e^{-kt} \)

0.75 = \( e^{-k(5 \text{ days})} \)

\[-k = \frac{\ln(0.75)}{5 \text{ days}} \Rightarrow k = 0.058 \text{ day}^{-1}\]

\[
t_{1/2} = \frac{\ln\left(\frac{1}{2}\right)}{-k} = \frac{0.692}{k}
\]

\[
t_{1/2} = \frac{0.692}{0.058 \text{ day}^{-1}} = 12 \text{ days}
\]
Disaggregated Growth

• More realistic cases involve several factors simultaneously, each with their own growth curves

→ Ex. Gasoline consumption depends on number of cars, miles driven, and fuel efficiency

Example

Examine total demand for lumber ($T$) for new home construction and assume all factors follow exponential growth.
$A = A_0 e^{bt} = \text{number of families needing homes}$

$C = C_0 e^{dt} = \text{square meters per home}$

$F = \text{constant} = \text{board meters needed per m}^2 \text{ of home space}$

$$T = F \times A \times C$$

$$T = F \times A_0 e^{bt} \times C_0 e^{dt}$$

$$T = F A_0 e^{bt} C_0 e^{dt}$$

$$T = F A_0 C_0 e^{bt + dt}$$

$$T = F A_0 C_0 e^{(b+d)t}$$
\[ T = FA_0C_0e^{(b+d)t} \]

Let \[ P_0 = FA_0C_0, \quad r = b + d \]

\[ P = P_0e^{rt} \]

For disaggregated systems,

- \( P_0 \) is the product of the constants,
- \( r \) is the sum of the rate coefficients
Resource Consumption

Carbon emissions are $5 \times 10^9$ tonne/yr and the atmosphere already has $700 \times 10^9$ tonnes. If emissions are growing at 4% per year, how long will it take to emit another $700 \times 10^9$ T?

With constant emission rate, this is easy:

$$\frac{700 \times 10^9 \text{ T}}{5 \times 10^9 \text{ T/yr}} = 140 \text{ yr}$$
Problem is not so easy if the emission rate is growing
— the production (as opposed to consumption) of a resource is increasing

Let $P$ be the production rate; $Q$ is the total resource produced between times $t_1$ and $t_2$

\[ P = P_0 e^{rt} \]
\[ Q = \int_{t_1}^{t_2} P \, dt = \int_{t_1}^{t_2} P_0 e^{rt} \, dt \]

$r$ is the growth rate of the production rate
Integrating over the times 0 to \( t \),

\[
Q = \int_0^t P_0 e^{rt} \, dt = \frac{P_0}{r} e^{rt} \bigg|_0^t = \frac{P_0}{r} (e^{rt} - 1)
\]

Take \ln{} of each side; rearrange:

\[
t = \left(\frac{1}{r}\right) \ln \left(\frac{rQ}{P_0} + 1\right)
\]

\( t \) is how long it takes to produce \( Q \) ...
How long does it take to produce 700 GT of C?

\[ t = \left( \frac{1}{r} \right) \ln \left( \frac{rQ}{P_0} + 1 \right) \]

\[ r = 4\% \text{ yr}^{-1} = 0.04 \text{ yr}^{-1} \]

\[ P_0 = 5 \times 10^9 \text{ T/yr} \]

\[ Q = 700 \times 10^9 \text{ T} \]

\[ t = 47 \text{ yr} \]
Other Growth Models

- Logistic Growth Curve
  → “Sigmoid” Growth—often applied to biological systems
• Gaussian Growth

→ Growth follows Gaussian curve—production falls in the future when resources become scarce

\[ P = P_m \exp \left[ -\frac{1}{2} \left( \frac{t - t_m}{\sigma} \right)^2 \right] \]

- \( P_m = \text{max. production rate} \)
- \( P = \text{production rate} \)
- \( t_m = \text{time } P_m \text{ occurs} \)
- \( \sigma = \text{standard deviation} \)
To get the total amount of resource ever produced:

\[ Q_\infty = \int_{-\infty}^{\infty} P \, dt = \int_{-\infty}^{\infty} P_m \exp \left[ -\frac{1}{2} \left( \frac{t - t_m}{\sigma} \right)^2 \right] dt \]

\[ = \sigma P_m \sqrt{2\pi} \]
Example (3.7)

Suppose the total production of U.S. coal is 4 times the 1997 recoverable reserves, estimated at 508 \times 10^9 \text{ tons}. If the U.S. production rate is 1.1 \times 10^9 \text{ tons/yr}, how long will it take to reach a peak production rate equal to 4 times the 1997 rate if a Gaussian production curve is followed?

First, solve for $\sigma$

$$\sigma = \frac{Q_\infty}{P_m \sqrt{2\pi}}$$

Then go back to equation for $P$ and set $t = 0$ to get $t_m$ in terms of $P_0$ and $P_m$ (eq. 3.20)
• Environmental impacts can be considered to be the product of human population and per capita consumption rates

• To predict future impacts on the environment, we need to know how the human population changes
• Crude Birth Rate: number of births per 1000 people per year.
  → Ranges ~ 10–40, depending on level of development

• Total Fertility Rate (TFR): number of births per woman during her lifetime

• Replacement Level Fertility (RLF): number of births needed to exactly replace each woman in the next generation
  → Ranges from 2.1 to 2.7, depending on infant mortality and that there may be unequal numbers of boys and girls
FIGURE 3.12 Annual births and total fertility rate in the United States.  
(Source: Bouvier and De Vita, 1991.)
Population Momentum
A continued increase in the population resulting from an earlier time when TFR was greater than RLF.

Crude Death Rates
Deaths per 1000 people per year. Highly variable

Infant Mortality Rate
Deaths before Age 1 per 1000 infants. Ranges ~ 10 to 150
Example

In 2006, 5.3 billion people lived in less-developed countries of the world, where the average crude birth and death rates were 23 and 8.5, respectively, and the infant mortality rate was 53.

What fraction of the total deaths was due to infant mortality?

Total deaths = Population \times \text{crude death rate}

= 5.3 \text{ billion people} \times \frac{8 \text{ deaths}}{1000 \text{ people}} = 42.4 \text{ million}
Infant deaths = Population \times \text{crude birth rate} \times \text{infant mortality rate}

= 5.3 \text{ billion people} \times \frac{23}{1000 \text{ people}} \times \frac{53}{1000 \text{ live births}}

= 6.5 \text{ million}

\text{Fraction infant deaths} = \frac{6.5 \text{ million}}{42.4 \text{ million}} = 0.15 = 15\%
Rate of Natural Increase \((r) = \text{crude birth rate (} b \text{) – crude death rate (} d \text{)}\)

**Table 3.3 Some Important Population Statistics (1995)**

<table>
<thead>
<tr>
<th></th>
<th>World</th>
<th>More developed countries</th>
<th>Less developed countries (non-China)</th>
<th>China</th>
<th>United States</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population (millions)</td>
<td>5702</td>
<td>1169</td>
<td>3314</td>
<td>1719</td>
<td>263</td>
</tr>
<tr>
<td>% of world population</td>
<td>100</td>
<td>20</td>
<td>58</td>
<td>22</td>
<td>4.6</td>
</tr>
<tr>
<td>Crude birth rate (b)</td>
<td>24</td>
<td>12</td>
<td>31</td>
<td>18</td>
<td>15</td>
</tr>
<tr>
<td>Crude death rate (d)</td>
<td>9</td>
<td>10</td>
<td>9</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>Natural increase (r) %</td>
<td>1.5</td>
<td>0.2</td>
<td>2.2</td>
<td>1.1</td>
<td>0.7</td>
</tr>
<tr>
<td>% Population</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>under age 15</td>
<td>32</td>
<td>20</td>
<td>38</td>
<td>27</td>
<td>22</td>
</tr>
<tr>
<td>Total fertility rate</td>
<td>3.1</td>
<td>1.6</td>
<td>4.0</td>
<td>1.9</td>
<td>2.0</td>
</tr>
<tr>
<td>Infant mortality rate</td>
<td>62</td>
<td>10</td>
<td>72</td>
<td>44</td>
<td>8</td>
</tr>
<tr>
<td>% of total added 1995–2025</td>
<td>100</td>
<td>4</td>
<td>85</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>Per capita GNP (US$)</td>
<td>4500</td>
<td>17270</td>
<td>1250</td>
<td>490</td>
<td>24750</td>
</tr>
<tr>
<td>% urban</td>
<td>43</td>
<td>74</td>
<td>38</td>
<td>28</td>
<td>75</td>
</tr>
<tr>
<td>Est. population</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2025 (millions)</td>
<td>8312</td>
<td>1271</td>
<td>5518</td>
<td>1523</td>
<td>358</td>
</tr>
<tr>
<td>Added pop. 1995–2025</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(millions)</td>
<td>2610</td>
<td>102</td>
<td>2204</td>
<td>304</td>
<td>75</td>
</tr>
</tbody>
</table>


For in individual country, we need to add or subtract the migration rate \((m)\) to \((b – d)\) to get \(r\)

\[
r = b - d + m
\]
Using U.S. numbers, the population is growing by about 2 million per year due to population momentum (immigration is 650000/yr legal + unknown number undocumented)

<table>
<thead>
<tr>
<th>COUNTRY</th>
<th>Total Population (Millions)</th>
<th>Birth Rate, per 1,000</th>
<th>Death Rate, per 1,000</th>
<th>Total Fertility Rate</th>
<th>Natural Increase (%)</th>
<th>Annual Growth Rate (%)</th>
<th>Doubling Time (Years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>China</td>
<td>1,256</td>
<td>14.5</td>
<td>7.0</td>
<td>1.79</td>
<td>0.76</td>
<td>0.71</td>
<td>97</td>
</tr>
<tr>
<td>India</td>
<td>1,018</td>
<td>24.9</td>
<td>8.3</td>
<td>3.11</td>
<td>1.65</td>
<td>1.65</td>
<td>42</td>
</tr>
<tr>
<td>United States</td>
<td>275</td>
<td>14.2</td>
<td>8.8</td>
<td>2.67</td>
<td>0.54</td>
<td>0.84</td>
<td>82</td>
</tr>
<tr>
<td>Brazil</td>
<td>174</td>
<td>19.9</td>
<td>9.1</td>
<td>2.23</td>
<td>1.09</td>
<td>1.08</td>
<td>64</td>
</tr>
<tr>
<td>Russia</td>
<td>146</td>
<td>9.7</td>
<td>15.0</td>
<td>1.33</td>
<td>-0.53</td>
<td>-0.34</td>
<td>—</td>
</tr>
<tr>
<td>Pakistan</td>
<td>141</td>
<td>32.6</td>
<td>10.2</td>
<td>4.56</td>
<td>2.24</td>
<td>2.15</td>
<td>32</td>
</tr>
<tr>
<td>Japan</td>
<td>126</td>
<td>10.6</td>
<td>8.3</td>
<td>1.50</td>
<td>0.23</td>
<td>0.20</td>
<td>345</td>
</tr>
<tr>
<td>Nigeria</td>
<td>117</td>
<td>41.4</td>
<td>13.0</td>
<td>5.95</td>
<td>2.84</td>
<td>2.87</td>
<td>24</td>
</tr>
<tr>
<td>Mexico</td>
<td>102</td>
<td>24.5</td>
<td>4.8</td>
<td>2.79</td>
<td>1.97</td>
<td>1.69</td>
<td>41</td>
</tr>
<tr>
<td>Germany</td>
<td>82</td>
<td>8.5</td>
<td>10.8</td>
<td>1.27</td>
<td>-0.23</td>
<td>-0.02</td>
<td>—</td>
</tr>
<tr>
<td>Ethiopia</td>
<td>61</td>
<td>44.0</td>
<td>21.6</td>
<td>6.75</td>
<td>2.23</td>
<td>2.11</td>
<td>33</td>
</tr>
<tr>
<td>France</td>
<td>59</td>
<td>11.1</td>
<td>9.2</td>
<td>1.58</td>
<td>0.19</td>
<td>0.23</td>
<td>300</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>59</td>
<td>11.8</td>
<td>10.6</td>
<td>1.72</td>
<td>0.12</td>
<td>0.22</td>
<td>313</td>
</tr>
<tr>
<td>Italy</td>
<td>57</td>
<td>9.4</td>
<td>10.4</td>
<td>1.25</td>
<td>-0.10</td>
<td>-0.09</td>
<td>—</td>
</tr>
<tr>
<td>Canada</td>
<td>31</td>
<td>11.6</td>
<td>7.3</td>
<td>1.64</td>
<td>0.43</td>
<td>1.02</td>
<td>68</td>
</tr>
<tr>
<td>Kenya</td>
<td>29</td>
<td>29.9</td>
<td>15.0</td>
<td>3.69</td>
<td>1.49</td>
<td>1.46</td>
<td>47</td>
</tr>
<tr>
<td>Sweden</td>
<td>8.9</td>
<td>11.7</td>
<td>11.0</td>
<td>1.78</td>
<td>0.07</td>
<td>0.27</td>
<td>259</td>
</tr>
</tbody>
</table>

SOURCE: Data from the U.S. Bureau of the Census International Data Base: http://www.census.gov/ftp/pub ipc/www/idbnew.html
Demographic Transition
Adjustment from the high birth rates (that nearly balanced high death rates before modern medicine) to a lower birth rate that balances the new death rates

**FIGURE 3.13** The demographic transition to low birth and death rates took over 100 years in the more developed nations. The less developed regions are in the middle of the process.
(Source: United Nations, 1971.)
Age Structures

Expansive population: Even if birth rate reached RLF, there will be continued population growth for many years

Near Stationary popnl.: This population has nearly achieved a constant level, and may even start to decline
US Population animation

http://www.ac.wwu.edu/~stephan/Animation/pyramid.html
China population animation

http://www.iiasa.ac.at/Research/LUC/ChinaFood/data/anim/pop_ani.htm

Note: Under the high fertility rate projection, which assumes that high fertility countries will stabilize at 2.6, and low fertility countries will rise to stabilize at 2.1, world population would reach 11.2 billion in 2050. Under the medium fertility rate projection, which assumes that the fertility rate ultimately will stabilize at a replacement level of about 2.1, the global population would reach about 9.4 billion is 2050. The low fertility rate projection assumes that countries currently with higher-than-replacement fertility rates will stabilize at 1.6, and that countries currently with lower-than-replacement rates will either stabilize at 1.5 or remain constant, under these assumptions, world population would stabilize at 7.7 billion in 2050.
Some Interesting Numbers

**A few energy numbers:**

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solar energy reaching Earth’s surface</td>
<td>$3 \times 10^{21}$ kJ/yr = $9.5 \times 10^{13}$ kW</td>
</tr>
<tr>
<td>US annual energy consumption</td>
<td>$10^{17}$ kJ</td>
</tr>
<tr>
<td>US per capita energy consumption</td>
<td>11 kW</td>
</tr>
</tbody>
</table>

**Per capita energy consumption:**

<table>
<thead>
<tr>
<th>Location</th>
<th>Per Capita Energy Consumption (kW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Germany</td>
<td>6 kW</td>
</tr>
<tr>
<td>Japan</td>
<td>5 kW</td>
</tr>
<tr>
<td>Global Average</td>
<td>2 kW</td>
</tr>
<tr>
<td>Developing world average</td>
<td>1 kW</td>
</tr>
</tbody>
</table>

Proven fossil fuel energy reserves: $4.3 \times 10^{19}$ kJ/yr

World’s annual energy consumption: $3.7 \times 10^{17}$ kJ/yr