

Mathematics of Growth and Human Population

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Growth and Population

- **Growth Rate**

- Exponential Growth
- Half-life and Doubling Times
- Disaggregated Growth

- **Resource Consumption**

- **Logistic and Gaussian Growth Models**

- **Human Population Growth**

- Birth, Death, Fertility Rates
- Age Structures

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Growth Rate

- **Growth rate enables prediction of future sizes—important for decisionmaking**

- Fuel usage and air pollution
- Improvements in energy efficiency
- Population growth and water demand
- Deforestation rates and global effects
- Cost and clean-up time of accidental contamination

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Exponential Growth

Growth rates are proportional to the present quantity of people, resources, etc.

Example: Number of students in a school increases by 2% each year.

N_0 = starting number of students

N_t = number of students in t years

r = annual growth rate

Year zero = N_0

Year one = $N_1 = N_0 + rN_0 = N_0(1 + r)$

Year two = $N_2 = N_1 + rN_1 = N_1(1 + r) = N_0(1 + r)^2$

Year three = $N_3 = N_2(1 + r) = N_0(1 + r)^3$

...Year $t = N_t = N_0(1 + r)^t$

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$$\text{Year } t = N_t = N_0(1 + r)^t$$

Exponential law for periodic increments of growth

— discrete increases at the end of each time period

Example

If the school has 1500 students now and the Board of Education decides to increase the student body by 2% every fall, how many students will there be in 7 years?

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$$\text{Year } t = N_t = N_0(1 + r)^t$$

$$t = 7 \text{ yr}$$

$$N_0 = 1500 \text{ students}$$

$$r = 0.02$$

$$\text{Year 7} = N_7 = 1500(1 + 0.02)^7$$

$$= 1723 \text{ students}$$

A more realistic model uses *continuous* growth over time, with the growth rate again proportional to population size N ...

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r is the growth rate with units 1/time

$$\frac{dN}{dt} = r \times N$$

$$N = N_0 e^{rt}$$

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Example

From 1990 to 1997, installed wind power in the U.S. grew by about 0.15% per year, resulting in a wind capacity of 1700 MW in 1997. If these rates are sustained, what will the wind energy capacity be in 2008?

$$N = N_0 e^{rt}$$

$$N_0 = 1700 \text{ MW}, r = 0.0015 \text{ yr}^{-1}, t = 11 \text{ years}$$

$$N = 1700 \text{ MW} \times e^{(0.0015 \text{ yr}^{-1} \cdot 11 \text{ yr})} = 1728 \text{ MW}$$

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Total Energy Consumption (U.S.)

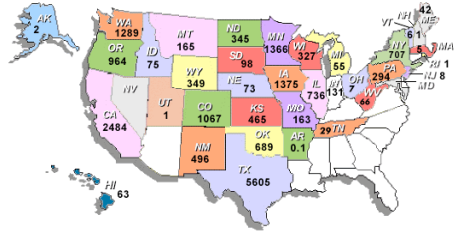
	2008	2035	2008–2035 growth rate
Petroleum (10^6 bbl/day)	20	22	0.4%
Natural Gas (10^{12} ft ³)	23.2	26.5	0.5%
Coal (10^{15} BTU)	22.4	24.3	0.3%
Nuclear (10^9 kW-hr)	806	874	0.3%
Hydropower and other renewables (10^{15} BTU)	7.0	11.8	1.9%

Source: www.eia.gov

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Actual growth of wind energy from 1998 to 2003 was about 23.1%, with total capacity in 2003 of 6374 MW. What is the new estimate of wind energy capacity in 2008?

$$N = 6374 \text{ MW} \times e^{(0.231 \text{ yr}^{-1} \cdot 5 \text{ yr})} = 20231 \text{ MW}$$



19549 MW this June; 9022 MW under construction

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Example

Decay instead of growth—

If the present mountain lion habitat in California is 8400 km², and is shrinking by 5% per year, when will it reach 5000 km²?

$$N = N_0 e^{rt}$$

$$5000 \text{ km}^2 = 8400 \text{ km}^2 e^{(-0.05 \text{ yr}^{-1})t}$$

$$t = \frac{\ln\left(\frac{5000}{8400}\right)}{-0.05 \text{ yr}^{-1}} = 10.4 \text{ yr}$$

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Doubling Times and Half-life

t_d = time for population to double

$t_{1/2}$ = time for half of population to decay away

$$N = N_0 e^{rt}$$

Grow from N_0 to $2N_0$:

$$2N_0 = N_0 e^{rt} \Rightarrow 2 = e^{rt} \Rightarrow \ln(2) = rt$$

$$t_d = \frac{\ln(2)}{r} = \frac{0.692}{r}$$

$$= \frac{70}{r(\%)}$$

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Example

If it took 300 years for the world's population to increase from 0.5 billion to 4 billion and we assume exponential growth over that time period, what is the growth rate?

$$N = N_0 e^{rt}$$

$$r = \frac{\ln\left(\frac{N}{N_0}\right)}{t}$$

$$r = 6.93 \times 10^{-3} \text{ yr}^{-1}$$

$$t_d = \frac{\ln(2)}{r} = \frac{0.692}{r}$$

$$0.5 \rightarrow 1.0 \rightarrow 2.0 \rightarrow 4.0$$

Took 300 years to double three times, so $t_d = 100 \text{ yr}$

$$r = 6.92 \times 10^{-3} \text{ yr}^{-1}$$

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Half-life

Switch to a decay situation where we go from N_0 to $0.5N_0$

$$N = N_0 e^{-kt}$$

$$N = \frac{1}{2} N_0 = N_0 e^{-kt}$$

$$t_{1/2} = \frac{\ln\left(\frac{1}{2}\right)}{-k} = \frac{\ln(1) - \ln(2)}{-k} = \frac{0.692}{k}$$

“Lifetime” is similar to half-life, except it’s how long it takes to decrease by 2.71 (e):

$$\tau = 1/k$$

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Example

A lake was impacted by a benzene spill of 100 kg. Benzene volatilizes and biodegrades, and after 5 days, there are 75 kg left.

When will half of the benzene be gone?

First find k , then get $t_{1/2}$

$$N = N_0 e^{-kt}$$

$$t_{1/2} = \frac{\ln\left(\frac{1}{2}\right)}{-k} = \frac{0.692}{k}$$

$$75 \text{ kg} = 100 \text{ kg} \cdot e^{-kt}$$

$$0.75 = e^{-k(5 \text{ days})}$$

$$t_{1/2} = \frac{0.692}{0.058 \text{ day}^{-1}}$$

$$-k = \frac{\ln(0.75)}{5 \text{ days}} \Rightarrow k = 0.058 \text{ day}^{-1}$$

$$= 12 \text{ days}$$

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Disaggregated Growth

- More realistic cases involve several factors simultaneously, each with their own growth curves

→ Ex. Gasoline consumption depends on number of cars, miles driven, and fuel efficiency

Example

Examine total demand for lumber (T) for new home construction and assume all factors follow exponential growth.

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$A = A_0 e^{bt}$ = number of families needing homes

$C = C_0 e^{dt}$ = square meters per home

F = constant = board meters needed per m² of home space

$$T = F \times A \times C$$

$$T = F \times A_0 e^{bt} \times C_0 e^{dt}$$

$$T = F A_0 e^{bt} C_0 e^{dt}$$

$$T = F A_0 C_0 e^{bt+dt}$$

$$T = F A_0 C_0 e^{(b+d)t}$$

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$$T = F A_0 C_0 e^{(b+d)t}$$

Let $P_0 = F A_0 C_0$, $r = b + d$

$$P = P_0 e^{rt}$$

For disaggregated systems,

P_0 is the product of the constants,

r is the sum of the rate coefficients

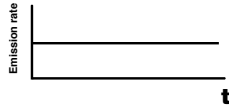
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Resource Consumption

Carbon emissions are 5×10^9 tonne/yr and the atmosphere already has 700×10^9 tonnes. If emissions are growing at 4% per year, how long will it take to emit another 700×10^9 T?

With constant emission rate, this is easy:

$$\frac{700 \times 10^9 \text{ T}}{5 \times 10^9 \text{ T/yr}} = 140 \text{ yr}$$



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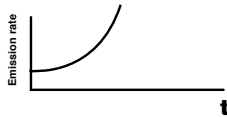
Problem is not so easy if the emission rate is growing

— the production (as opposed to consumption) of a resource is increasing

Let P be the production rate; Q is the total resource produced between times t_1 and t_2

$$P = P_0 e^{rt}$$

$$Q = \int_{t_1}^{t_2} P \, dt = \int_{t_1}^{t_2} P_0 e^{rt} \, dt$$



r is the growth rate of the production rate

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Integrating over the times 0 to t ,

$$Q = \int_0^t P_0 e^{rt} \, dt = \frac{P_0}{r} e^{rt} \Big|_0^t = \frac{P_0}{r} (e^{rt} - 1)$$

Take $\ln$ of each side; rearrange:

$$t = \left(\frac{1}{r} \right) \ln \left(\frac{rQ}{P_0} + 1 \right)$$

t is how long it takes to produce Q ...

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How long does it take to produce 700 GT of C?

$$t = \left(\frac{1}{r}\right) \ln\left(\frac{rQ}{P_0} + 1\right)$$

$$r = 4\% \text{ yr}^{-1} = 0.04 \text{ yr}^{-1}$$

$$P_0 = 5 \times 10^9 \text{ T/yr}$$

$$Q = 700 \times 10^9 \text{ T}$$

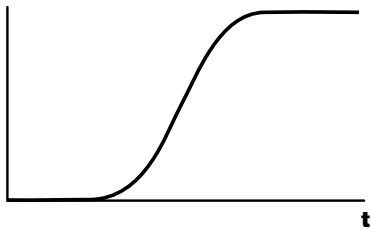
$$t = 47 \text{ yr}$$

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Other Growth Models

• Logistic Growth Curve

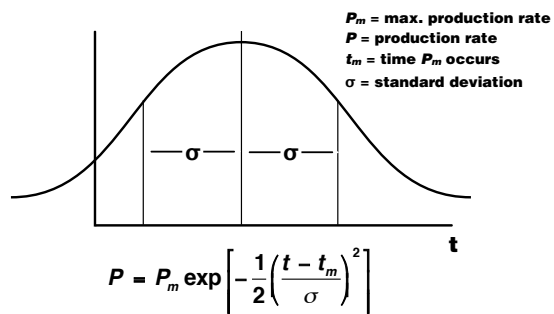
→ “Sigmoid” Growth—often applied to biological systems



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• Gaussian Growth

→ Growth follows Gaussian curve—production falls in the future when resources become scarce



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To get the total amount of resource ever produced:

$$Q_{\infty} = \int_{-\infty}^{\infty} P \, dt = \int_{-\infty}^{\infty} P_m \exp \left[-\frac{1}{2} \left(\frac{t - t_m}{\sigma} \right)^2 \right] dt$$

$$= \sigma P_m \sqrt{2\pi}$$

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Example (3.7)

Suppose the total production of U.S. coal is 4 times the 1997 recoverable reserves, estimated at 508×10^9 tons. If the U.S. production rate is 1.1×10^9 tons/yr, how long will it take to reach a peak production rate equal to 4 times the 1997 rate if a Gaussian production curve is followed?

First, solve for σ

$$\sigma = \frac{Q_{\infty}}{P_m \sqrt{2\pi}}$$

Then go back to equation for P and set $t = 0$ to get t_m in terms of P_0 and P_m (eq. 3.20)

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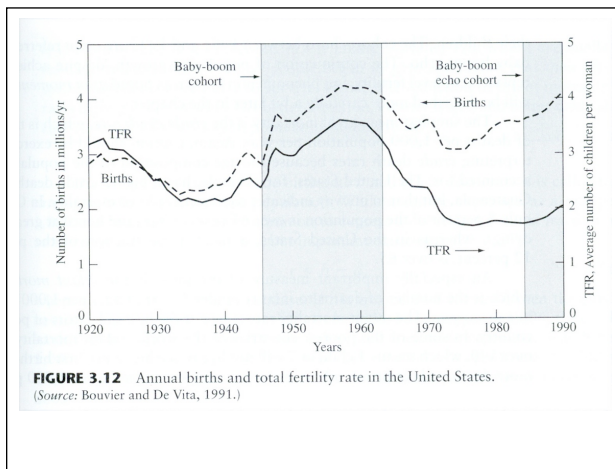
Human Populations

- **Environmental impacts can be considered to be the product of human population and per capita consumption rates**
- **To predict future impacts on the environment, we need to know how the human population changes**

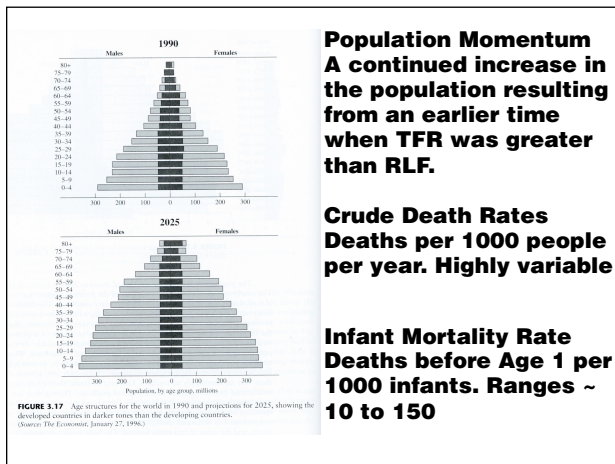
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- **Crude Birth Rate: number of births per 1000 people per year.**
 - **Ranges ~ 10–40, depending on level of development**
- **Total Fertility Rate (TFR): number of births per woman during her lifetime**
- **Replacement Level Fertility (RLF): number of births needed to exactly replace each woman in the next generation**
 - **Ranges from 2.1 to 2.7, depending on infant mortality and that there may be unequal numbers of boys and girls**

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Example

In 2006, 5.3 billion people lived in less-developed countries of the world, where the average crude birth and death rates were 23 and 8.5, respectively, and the infant mortality rate was 53.

What fraction of the total deaths was due to infant mortality?

Total deaths = Population $\times$ crude death rate

$$= 5.3 \text{ billion people} \times \frac{8.5 \text{ deaths}}{1000 \text{ people}} = 42.4 \text{ million}$$

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Infant deaths = Population $\times$ crude birth rate $\times$ infant mortality rate

$$= 5.3 \text{ billion people} \times \frac{23}{1000 \text{ people}} \times \frac{53}{1000 \text{ live births}}$$

$$= 6.5 \text{ million}$$

$$\text{Fraction infant deaths} = \frac{6.5 \text{ million}}{42.4 \text{ million}} = 0.15 = 15\%$$

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Rate of Natural Increase (r) = crude birth rate (b) – crude death rate (d)

TABLE 3.3 Some Important Population Statistics (1995)

	World	More developed countries	Less developed countries (non-China)	China	United States
Population (millions)	5702	1169	3314	1210	263
% of world population	100	20	58	22	4.6
Crude birth rate b	24	12	31	18	15
Crude death rate d	9	10	9	6	9
Natural increase r %	1.5	0.2	2.2	1.1	0.7
% Population under age 15	32	20	38	27	22
Total fertility rate	3.1	1.6	4.0	1.9	2.0
Infant mortality rate	62	10	72	44	8
% of total added 1995–2025	100	4	85	11	3
Per capita GNP (US\$)	4500	17270	1250	490	24750
% urban	43	74	38	28	75
Est. population 2025 (millions)	8312	1271	5518	1523	358
Added pop. 1995–2025 (millions)	2610	102	2204	304	75

Source: Population Reference Bureau, 1996.

For in individual country, we need to add or subtract the migration rate (m) to ($b - d$) to get r

$$r = b - d + m$$

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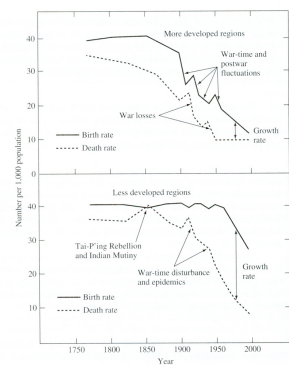
Using U.S. numbers, the population is growing by about 2 million per year due to population momentum (immigration is 650000/yr legal + unknown number undocumented)

TABLE 6.2 Year 2000 Population Statistics for Selected Countries

COUNTRY	Total Population (Millions)	Birth Rate, per 1,000	Death Rate, per 1,000	Total Fertility Rate	Natural Increase (%)	Annual Growth Rate (%)	Doubling Time (Years)
China	1,256	14.5	7.0	1.79	0.76	0.71	97
India	1,018	24.9	8.3	3.11	1.65	1.65	42
United States	275	14.2	8.8	2.07	0.54	0.84	82
Brazil	174	19.9	9.1	2.23	1.09	1.08	64
Russia	146	9.7	15.0	1.33	-0.53	-0.34	—
Pakistan	141	32.6	10.2	4.56	2.24	2.15	32
Japan	126	10.6	8.3	1.50	0.23	0.20	345
Nigeria	117	41.4	15.0	5.95	2.84	2.87	24
Mexico	102	24.5	4.8	2.79	1.97	1.69	41
Germany	82	8.5	10.8	1.27	-0.23	-0.02	—
Ethiopia	61	44.0	21.6	6.75	2.23	2.11	33
France	59	11.1	9.2	1.58	0.19	0.23	200
United Kingdom	59	11.8	10.6	1.72	0.12	0.22	313
Italy	57	9.4	10.4	1.25	-0.10	-0.09	—
Canada	31	11.6	7.3	1.64	0.63	1.02	68
Kenya	29	29.9	15.0	3.69	1.49	1.46	47
Sweden	8.9	11.7	11.0	1.78	0.07	0.27	259

SOURCE: Data from the U.S. Bureau of the Census, International Data Base <http://www.census.gov/hip/pub/pc/www/indmex.html>

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Demographic Transition Adjustment from the high birth rates (that nearly balanced high death rates before modern medicine) to a lower birth rate that balances the new death rates

FIGURE 3.13 The demographic transition to low birth and death rates took over 100 years in the more developed nations. The less developed regions are in the middle of the process. (Source: United Nations, 1971.)

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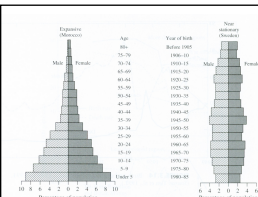


FIGURE 3.15 A rapidly growing, expansive population (Mormons), compared with a population that is nearing stationary condition (Sweden). (Source: Hargis and Jones, 1982.)

Age Structures

Expansive population: Even if birth rate reached RLF, there will be continued population growth for many years

Near Stationary popl.: This population has nearly achieved a constant level, and may even start to decline

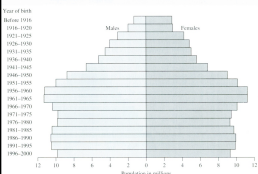
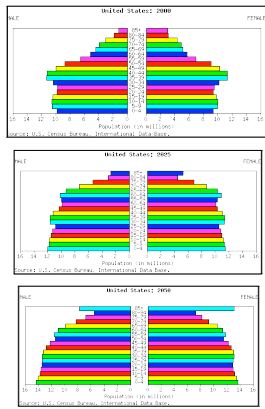


FIGURE 3.16 The age structure for the United States in 2000. (Source: U.S. Census Bureau.)

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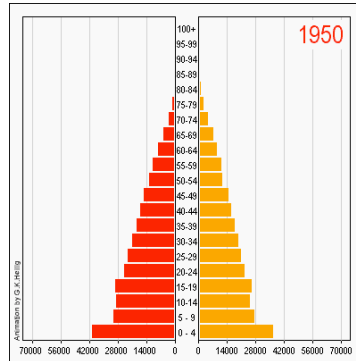
US Population animation



<http://www.ac.wvu.edu/~stephan/Animation/pyramid.html>

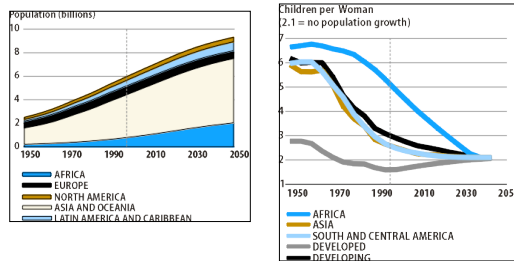
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China population animation

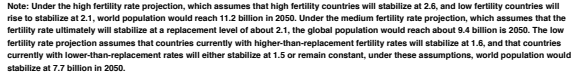


http://www.iiasa.ac.at/Research/LUC/ChinaFood/data/anim/pop_ani.htm

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Some Interesting Numbers

A few energy numbers:

Solar energy reaching Earth's surface	3×10^{21} kJ/yr = 9.5×10^{13} kW
US annual energy consumption	10^{17} kJ
US per capita energy consumption	11 kW

Per capita energy consumption:

Germany	6 kW
Japan	5 kW
Global Average	2 kW
Developing world average	1 kW

Proven fossil fuel energy reserves:	4.3×10^{19} kJ/yr
World's annual energy consumption:	3.7×10^{17} kJ/yr